Lab 8: Linear Programming

Lab date: November 6 Due date: November 12

Goals: In this lab we will write a code to solve a linear programming problem where there are a small number of constraints; we will use the approach of finding all feasible solutions and evaluating our objective function there. Our method will not be as efficient as the Simplex Method but it should be straightforward to program and it works well for small problems. Once we get our code working we will use it to solve a standard problem in satisfying supply and demand.

Recall that the matrix form of Linear Programming problem is: vector $x = (x_1, x_2, ..., x_n)^T$ which is subject to the constraints

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{1}$$

$$x_j \geq 0, \qquad j = 1, 2, 3, 4, \ldots \tag{2}$$

Because inequality constraints are difficult to solve we introduce slack variables to get the following problem. Let $c = (c1, c2, , cn, 0, 0, ..., 0) \in R^s$ where $s = n + m$. Then we see $x \in R^s$ to

$$\text{subject to the equality constraints } Ax = b, \quad x \geq 0 \tag{3}$$

Here A is the $m \times s$ rank m matrix with $m \leq s$ given by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \ldots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & 0 & \vdots & 1 \end{pmatrix} \tag{4}$$

In the lecture we saw that to solve a Linear Programming problem with m inequality constraints defining a bounded feasible region we can follow these steps.

- find all basic solutions by solving all possible $m \times m$ linear systems;
- discard the ones which are not feasible;
- evaluate the function to maximize at each feasible basic solution and pick the maximum.

In the first exercise we write a code which implements this approach and then in the second we apply it to a problem in satisfying supply and demand.

1. Write a code which implements the above approach for solving our Linear Programming problem where we have introduced slack variables. Your code should have the following structure.
   Input: the under-determined matrix A; the vector b giving the right hand side of the constraints; the vector c where we want to maximize $z = c^T x$.
   Output: your code should print out each solution, whether it is feasible or not; if it is feasible it should print the value of the objective function there. Your code should return the location of the maximum and its value there.

You can simply use the “backslash command to solve your linear systems. There may be some reduced matrices which are singular so you should add a conditional to only solve linear systems that are full rank.
You should try to write your code in a general way; i.e., it should work whether your under-determined system is 2×4 or 7×13. Test your code on our example from class: seek x such that x maximizes

$$x = \begin{pmatrix} 120 & 100 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5}$$

subject to

$$2x1 + 2x_2 \leq 8, \qquad 5x_1 + 3x_2 \leq 15, \qquad x_1, x_2 \leq 15 \tag{6}$$

We converted this using slack variables x_3, x_4 to get the under-determined system

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

2. The environmentally conscious CEO of the Green Sawmill Company wants to make sure she keeps a steady but sufficient flow of logs (which were harvested from planted hardwoods by her grandfather) to the sawmills to capitalize on the good lumber market. Of course, she would like to maximize profits and the way she can do this is to keep her transportation costs as low as possible. Taking advantage of the skills she learned at FSU, she decides to pose her problem as a linear programming problem and then solve it. There are three logging sites and three mills and the distance in miles between each is given in the table below.

Logging site	Distance to Mill #1	Distance to Mill #2	Distance to Mill #3
#1	8	15	50
#2	10	17	20
#3	30	26	15

The average haul cost is $2 per mile for both loaded and empty trucks. Due to the size of each crew there is a maximum number of trees that can be cut and loaded from each site. For Site #1 this is 20 truckloads for day;. for Site #2 this is 30 truckloads for day and for Site #3 this is 45 truckloads for day. Finally, each mill has told the CEO that in order to handle her trees she must provide a minimum number of truckloads per day. Mill #1 sets this at 30 truckloads; Mill #2 sets this at 35 truckloads and Mill #3 sets this at 30 truckloads.

(a) Set this up as a linear programming problem with inequality constraints where you let x_{ij} denote the hauling costs from Site # i to Mill # j.

(b) Introduce slack variables to transform this problem into one with equality constraints.

(c) Use your code in # 1 to solve this problem. Output all feasible solutions and the corresponding transportation costs. Make your recommendation on shipping based on your solution.